I. AVERAGE BEAM-BEAM KICK

Assume that both beams (strong and weak) are Gaussian beams, but not round. Denote the rms beam sizes by (σ_x, σ_y) for the strong beam and $(\bar{\sigma}_x, \bar{\sigma}_y)$ for the weak beam. The kick felt by a particle in the weak beam with betatron amplitudes (x, y) is given by

$$\Delta x' = -\frac{2N_b r_p (x_p - x)}{\gamma_p} \int_0^\infty \frac{dq}{\left(2\sigma_x^2 + q\right)^{3/2} \left(2\sigma_y^2 + q\right)^{1/2}} \exp\left(-\frac{(x_p - x)^2}{2\sigma_x^2 + q} - \frac{(y_p - y)^2}{2\sigma_y^2 + q}\right), \tag{1}$$

$$\Delta y' = -\frac{2N_b r_p \left(y_p - y\right)}{\gamma_p} \int_0^\infty \frac{dq}{\left(2\sigma_x^2 + q\right)^{1/2} \left(2\sigma_y^2 + q\right)^{3/2}} \exp\left(-\frac{x^2}{2\sigma_x^2 + q} - \frac{y^2}{2\sigma_y^2 + q}\right), \tag{2}$$

where (x_p, y_p) denotes the location of the strong beam centroid w.r.t. the closed orbit of the weak beam.

The density distribution function of the weak beam is given by

$$\rho\left(x, y\right) = \frac{1}{2\pi\bar{\sigma}_x\bar{\sigma}_y} \exp\left(-\frac{x^2}{2\bar{\sigma}_x^2} - \frac{y^2}{2\bar{\sigma}_y^2}\right). \tag{3}$$

Therefore, the average kick given to the weak beam is obtained as

$$\langle \Delta x' \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \Delta x' \rho\left(x, y\right), \tag{4}$$

$$\langle \Delta y' \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \Delta y' \rho\left(x, y\right). \tag{5}$$

Due to the smoothness of the integrand we can change the order of the integration, and write the relations as

$$\langle \Delta x' \rangle = -\frac{2N_b r_p}{\gamma_p} \frac{1}{2\pi\bar{\sigma}_x\bar{\sigma}_y} \int_0^\infty \frac{dq}{\left(2\sigma_x^2 + q\right)^{3/2} \left(2\sigma_y^2 + q\right)^{1/2}} I_x\left(q\right) I_y\left(q\right), \tag{6}$$

where

$$I_x\left(q\right) = \int_{-\infty}^{\infty} dx \left(x_p - x\right) \exp\left(-\frac{(x_p - x)^2}{2\sigma_x^2 + q} - \frac{x^2}{2\bar{\sigma}_x^2}\right), \tag{7}$$

$$I_y\left(q\right) = \int_{-\infty}^{\infty} dy \exp\left(-\frac{(y_p - y)^2}{2\sigma_y^2 + q} - \frac{y^2}{2\bar{\sigma}_y^2}\right). \tag{8}$$

The integrations can be done analytically, and give the following results:

$$I_x\left(q\right) = \frac{\sqrt{2\pi}\bar{\sigma}_x \left(2\sigma_x^2 + q\right)^{3/2} x_p}{\left[2\left(\sigma_x^2 + \bar{\sigma}_x^2\right) + q\right]^{3/2}} \exp\left[-\frac{x_p^2}{2\left(\sigma_x^2 + \bar{\sigma}_x^2\right) + q}\right], \tag{9}$$

$$I_y\left(q\right) = \frac{\sqrt{2\pi}\bar{\sigma}_y \left(2\sigma_y^2 + q\right)^{1/2}}{\left[2\left(\sigma_y^2 + \bar{\sigma}_y^2\right) + q\right]^{1/2}} \exp\left[-\frac{y_p^2}{2\left(\sigma_y^2 + \bar{\sigma}_y^2\right) + q}\right]. \tag{10}$$

Hence

$$\langle \Delta x' \rangle = -\frac{2N_b r_p x_p}{\gamma_p} \int_0^\infty \frac{dq}{\left[2\left(\sigma_x^2 + \bar{\sigma}_x^2\right) + q\right]^{3/2} \left[2\left(\sigma_y^2 + \bar{\sigma}_y^2\right) + q\right]^{1/2}} \exp\left[-\frac{x_p^2}{2\left(\sigma_x^2 + \bar{\sigma}_x^2\right) + q} - \frac{y_p^2}{2\left(\sigma_y^2 + \bar{\sigma}_y^2\right) + q}\right], \tag{11}$$

that is, $\langle \Delta x' \rangle$ is the kick felt by the weak beam's centroid from a beam with an effective beam size being equal with the rms beam size of the strong and weak beams. Due to symmetry in x and y, $\langle \Delta y' \rangle$ looks similar, i.e. $\langle \Delta x' \rangle$ with x and y interchanged. Of course, the numerical evaluation of the average kick is done more efficiently utilizing the well-known Bassetti-Erskine formulae.

A. Special case of round beams

To check the result, the special case of round beams can be studied, where all integrals can be performed analytically. The single particle kick in this case is given by ($\sigma = \sigma_x = \sigma_y$)

$$\Delta x' = -\frac{2N_b r_p}{\gamma_p} \frac{x_p - x}{(x_p - x)^2 + (y_p - y)^2} \left[1 - \exp\left(-\frac{(x_p - x)^2 + (y_p - y)^2}{2\sigma^2}\right)\right], \tag{12}$$

$$\Delta y' = -\frac{2N_b r_p}{\gamma_p} \frac{y_p - y}{(x_p - x)^2 + (y_p - y)^2} \left[1 - \exp\left(-\frac{(x_p - x)^2 + (y_p - y)^2}{2\sigma^2}\right)\right]. \tag{13}$$

The integrals in this case cannot be separated into integrations over x and y independently, so we switch to polar coordinates using

$$x_p - x = r\cos\phi, \tag{14}$$

$$y_p - y = r\sin\phi. \tag{15}$$

The average kick becomes ($\bar{\sigma} = \bar{\sigma}_x = \bar{\sigma}_y$)

$$\langle \Delta x' \rangle = -\frac{2N_b r_p}{\gamma_p} \frac{1}{2\pi\bar{\sigma}^2} \exp\left(-\frac{x_p^2 + y_p^2}{2\bar{\sigma}^2}\right) \int_0^\infty dr \left[1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right] \exp\left(-\frac{r^2}{2\bar{\sigma}^2}\right) \tag{16}$$

$$\times \int_0^{2\pi} d\phi \exp\left[\frac{r\left(x_p\cos\phi + y_p\sin\phi\right)}{\bar{\sigma}^2}\right]\cos\phi. \tag{17}$$

The integration over ϕ is given by (from Integrals & Series, vol.1, page 464)

$$\int_0^{2\pi} d\phi \exp\left[\frac{r\left(x_p\cos\phi + y_p\sin\phi\right)}{\bar{\sigma}^2}\right]\cos\phi = \frac{2\pi x_p}{\sqrt{x_p^2 + y_p^2}} I_1\left(\frac{r\sqrt{x_p^2 + y_p^2}}{\bar{\sigma}^2}\right), \tag{18}$$

where $I_1(z)$ is the modified Bessel function. If we denote by $d = \sqrt{x_p^2 + y_p^2}$, the last integral to be done is

$$\int_0^\infty dr \left[1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right] \exp\left(-\frac{r^2}{2\bar{\sigma}^2}\right) I_1\left(\frac{d}{\bar{\sigma}^2} r\right). \tag{19}$$

Mathematica gives the result

$$\frac{\bar{\sigma}^2}{d}\left\{\exp\left(\frac{d^2}{2\bar{\sigma}^2}\right) - \exp\left[\frac{d^2\sigma^2}{2\bar{\sigma}^2\left(\sigma^2 + \bar{\sigma}^2\right)}\right]\right\}. \tag{20}$$

Combining everything we recover the special case result

$$\langle \Delta x' \rangle = -\frac{2N_b r_p}{\gamma_p} \frac{x_p}{d^2}\left\{1 - \exp\left[-\frac{d^2}{2\left(\sigma^2 + \bar{\sigma}^2\right)}\right]\right\}, \tag{21}$$

and analogously for $\langle \Delta y' \rangle$.